Decibels

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Engineers and technicians are often confronted with calculating or measuring characteristics of signal transmission systems, including power of a signal, power loss in a circuit, gain of an amplifier, or the sensitivity of a detector. The signals involved might be electrical, optical, radio, or acoustic. These quantities all require determining *ratios* of two numbers— the ratio of an output power to an input power, or the ratio of a power level to a standard unit of power.

Power is measured in *watts.* The output power P_0 watts at a given point on a transmission line is seen to be the input power P_i times a loss factor L

$$P_o = L\,P_i \qquad \text{Eq. 1}$$

L depends on the length of the line and technical factors about the line. In a passive transmission line, L is less than 1. If the line includes an amplifier or regenerator, P_0 could be greater then 1. L is the ratio of P_o to P_i.

$$L = \frac{P_o}{P_i} \qquad \text{Eq. 2}$$

Being the ratio of two powers, L is dimensionless, of dimension 1. These ratios can take on very large to very small values, and would often require repeated multiplication or division to obtain an overall result for a system. Performing these multiplications and divisions is unwieldy.

Addition and subtraction are much easier than multiplication and division. In 1614, the Scottish mathematician John Napier[1] invented a method of calculation that turns multiplication into addition and division into subtraction — the *logarithm*. Logarithms were the single most important improvement in arithmetic calculation before the modern computer and handheld digital calculator. What made them so useful is their ability to reduce multiplication to addition and division to subtraction.

In 1924, engineers at Bell Telephone Laboratories adopted the logarithm to define a unit for signal loss in telephone lines, the *transmission unit* (TU). The TU replaced the earlier standard unit, *miles of standard cable* (MSC), which had been in place since the introduction of telephone cable in 1896. 1 MSC corresponded to the loss of signal power over 1 mile of standard cable. Standard cable was defined as having a resistance of 88 ohms and capacitance of 0.054 microfarads per mile. 1 MSC equals 1.056 TU. The loss factor in TU was ten times the base-10 logarithm of the ratio of the output power to the input power.

In 1928, Bell Telephone Laboratories renamed the transmission unit the *decibel* (dB). The prefix 'deci' comes from Latin *decimus* 'tenth'. A decibel is one tenth of a *bel* (B), the unit named in honor of Alexander Graham Bell, inventor of the telephone

[1] Napier also invented the use of the decimal point to denote fractions.

in 1879 and founder, in 1885, of the American Telephone and Telegraph Company (AT&T). The bel is rarely used; the decibel has become widely used. One decibel is about the smallest attenuation detectable by an average listener, and corresponds to a signal power loss of 20.6%.

Interestingly, the smallest detectable change in sound level by listeners is relatively independent of the level – about 1 dB at any level, 20.6% power reduction. This means that human perception of loudness is logarithmic. It is the percentage change in level that matters, not the absolute change in watts. Quantities whose significance is proportional to a constant percent change are logarithmic, e.g., compound interest.

Note that the loss factor L is *not* the power loss. L is the ratio of two values of power, a dimensionless quantity, a number. The power loss itself is $P_o - P_i$, which has units of watts. By convention, a negative difference represents a power loss; a positive difference represents a power gain, as with an amplifier.

The power loss (or gain) can be expressed as a fraction of the input power

$$\frac{P_o - P_i}{P_i} = \frac{P_o}{P_i} - 1 = L - 1 \qquad \text{Eq. 3}$$

The loss factor in decibels L_{dB} is *defined* to be 10 times the base-10 logarithm of L.

$$L_{dB} \triangleq 10 \log_{10} L = 10 \log_{10} \frac{P_o}{P_i} \qquad \text{Eq. 4}$$

The quantity of real interest is the loss factor L. Decibel is just a convenient unit in which to represent L.

Absolute vs. Relative Power Levels

When measuring loss or gain, it is customary to set P_i to an arbitrary *reference level*, measure P_o and determine the ratio L and L_{dB}. Test instruments do the math. The practical procedure is to connect a reference source to the instrument, note its level as the reference level P_i then connect the output P_o to the instrument and read the gain or loss in decibels from the instrument.

To measure absolute power levels, the test instrument must be calibrated to an international standard unit of power, typically 1 milliwatt, 0.001 watt. Calibration is first performed when the instrument is manufactured and periodically thereafter. These calibrations are traceable to the international standard watt using a transfer standard maintained by a national laboratory. In the US, this laboratory is NIST, the National Institute of Science and Technology of the Department of Commerce.

Setting the instrument to measure absolute power causes it to use its calibrated reference level of $P_i = 0.001$ watt. The procedure is simpler: there is no need to measure the reference source, only the output, P_o.

To distinguish an absolute power level in decibels relative to one milliwatt, the unit symbol **dBm** is used. The unit symbol **dB** is used for relative power measurements where the reference power level is unspecified.

Decibels for Field Values

A decibel quantity corresponds to a *power ratio*, i.e. the ratio of two power levels. Sometimes instruments measure voltage or current in an electrical circuit, or electric or magnetic field strength in other applications, not power. To get a decibel value from a voltage, current, or field level ratio that is the same as if power were measured, Eq. 4 needs to be adjusted.

It turns out that power is proportional to the *square* of voltage, current, or field levels. Doubling the voltage quadruples the power. In this case, $L_{field} = V_o/V_i$ is used and Eq. 4 becomes

$$L_{dB} \triangleq 20\log_{10} L_{field} = 20\log_{10}\frac{V_o}{V_i} = 10\log_{10}\left(\frac{V_o}{V_i}\right)^2 \qquad \text{Eq. 5}$$

Logarithms

To understand the decibel, it is necessary first to understand logarithms. The mathematics used here is taught in high school. The presentation is heuristic, starting with counting and elementary multiplication and division with integers, then building to include rational numbers and, finally, all numbers.

The function $f(x) = a^x$ is called the *exponential function with base a*, *a > 0*.
Note that when $a = 1, f(x) = 1$ for all *x*, so $a = 1$ is generally excluded.

Consider a positive number *a* multiplied by itself *n* times. *n* is thus a positive integer. Let *x* = *n*. Then $f(n) = a^n = a * a * a \ldots a$, i.e. *a* repeated *n* times.

The *logarithm of* a^n *of base-a* is *defined* as the exponent *n*.

$$\log_a a^n \triangleq n \qquad \text{Eq. 6}$$

$$a^{\log_a a^n} = a^n \qquad \text{Eq. 7}$$

A logarithm is an exponent. Eq. 7 shows that the logarithm function is the inverse of the exponential function.

Suppose we have another exponential with base *a* having *m* factors, a^m, where *m* is also a positive integer. If we form the product $a^n\ a^m$, then we have $n + m$ repetitions of *a.* However, this longer product is the same as a^{n+m}. Thus, we have

$$a^n a^m = a^{n+m} \qquad \text{Eq. 8}$$

$$\log_a(a^n a^m) = \log_a a^{n+m} = n + m = \log_a a^n + \log_a a^m \qquad \text{Eq. 9}$$

Here we have in Napier's invention a way to represent the product of two numbers $a^n a^m$ as the sum of logarithms $n + m$ of those two numbers. When the numbers are represented as exponentials with a common base *a*, we add their logarithms.

Suppose *n* is greater than *m* and we divide instead of multiply. Then we have

$$\frac{a^n}{a^m} = a^{n-m} \qquad \text{Eq. 10}$$

This is so because the m as in the denominator cancel m as in the numerator, leaving $n - m$ as the total number of as.

$$\log_a \frac{a^n}{a^m} = \log_a a^{n-m} = n - m = \log_a a^n - \log_a a^m \qquad \text{Eq. 11}$$

Here we likewise have a way to represent the quotient of two numbers a^n/a^m as the difference of logarithms $n - m$ of those two numbers. When the numbers are represented as exponentials with a common base a, we subtract their logarithms.

Note that $a^0 = 1$, for any a, and $a^{n-n} = a^n \, a^{-n} = a^0 = 1$, so $a^{-n} = \frac{1}{a^n} = \frac{a^0}{a^n}$. Then

$$\log_a 1 = 0, \qquad \text{for all } a \qquad \text{Eq. 12}$$

$$\log_a a^{-n} = \log_a \frac{1}{a^n} = \log_a \frac{a^0}{a^n} = 0 - n = -\log_a a^n \qquad \text{Eq. 13}$$

The logarithm of the reciprocal is the negative of the logarithm.

So far, we have shown how to calculate the logarithm of any integer power of a and the logarithm of the reciprocal of any integer power of a, where a can be any positive number, not restricted to integers, just $a > 0$. a cannot be zero, since $0^n = 0$ and division by zero is undefined.

We would like to represent *any* number as a chosen base raised to some power. It turns out this can be done for all positive numbers. However, so far, we have only shown that this works when n and m are positive integers. We can expand the domain of x in the exponential function by showing how to calculate $f(x)$ when x is a *rational number*, i.e. the ratio of two integers.

Consider $a^{\frac{1}{m}}$, which is defined to be the number that, when multiplied by itself m times, gives a, i.e. $a^{\frac{1}{m}}$ is the m^{th} *root* of a. When m=2, $a^{\frac{1}{2}}$ is the square root; when m=3, $a^{\frac{1}{3}}$ is the cube root, and so forth. By multiplying $a^{\frac{1}{m}}$ by itself n times

$$\left(a^{\frac{1}{m}}\right)^n = a^{\frac{n}{m}} \qquad \text{Eq. 14}$$

$$\log_a a^{\frac{n}{m}} = \frac{n}{m} \qquad \text{Eq. 15}$$

We now have a way to calculate the exponential $f(x)$ for x any *rational* number. This also works for *irrational numbers*, numbers that cannot be expressed as the ratio of two integers, e.g. $\pi = 3.141592 \ldots$, where the decimals never repeat. Since rounding an irrational number to a fixed number of decimal places always results in a rational number, extending the number of decimal places indefinitely also works.

The logarithm function with base a is $y = \log_a x$. It is defined as the inverse of the exponential function with base a, $y = a^x$ $(\mathrm{a} > 0, \mathrm{a} \neq 1)$.

The domain of $\log_a x$ is $(0, \infty)$, which is the range of a^x.
The range of $\log_a x$ is $(-\infty, \infty)$, which is the domain of a^x.
Eq. 16 shows this inverse relationship.

$$\log\left(a^{\log_a a^x}\right) = \log_a a^x = x\,, \qquad a > 0, a \neq 1, x > 0 \qquad \text{Eq. 16}$$

The *domain* of a function $y = f(x)$ is the set of values of x for which the function is defined. The *range* of the function is the set of values of the function, y. The domain and range of the exponential and logarithm functions are *open intervals*, i.e. they do not include the endpoints 0 or $\pm\infty$. Another example: $y = x^2$ and $y = \sqrt{x}$ are inverses. The domain of one is the range of the other. $\left(\sqrt{x}\right)^2 = \sqrt{x^2} = x, x \geq 0$.

Summary of the Rules for Exponentials

If $a > 0$ and $b > 0$, the following rules hold true for all real numbers x and y.

1. $a^x\, a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x b^x = (ab)^x$
5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

Summary of the Rules for Logarithms

For any numbers $a > 0$, $a \neq 1$, $b > 0$ and $x > 0$, the logarithm of base-a function satisfies the following rules:

1. *Product Rule*: $\log_a bx = \log_a b + \log_a x$
2. *Quotient Rule:* $\log_a \frac{b}{x} = \log_a b - \log_a x$
3. *Reciprocal Rule:* $\log_a \frac{1}{x} = -\log_a x$
4. *Power Rule:* $\log_a x^r = \mathrm{r} \log_a x$
5. *Conversion Rule:* $\log_b x = \log_\mathrm{a} x\, \log_b a$

Applications

Three values of the logarithm base, a, are widely used: 10, 2, and e = 2.71828….
Ten is used for decibels. Two is used in computer science. Since a binary number comprising n bits can take on 2^n possible values, the number of bits required to represent a given positive integer N is $n = \log_2 N$, rounded up to the next bit.
e is *Euler's Number*, a value of particular importance in calculus. e^x and all of its derivatives are the same, e^x. Euler's Number also appears in the fascinating equation $e^{i\pi} - 1 = 0$, where $i^2 = -1$. This equation, also due to Leonard Euler (1707-1783), relates five of the most important constants in mathematics.

Logarithms of base-10 are called *common logarithms,* commonly written as "log x." Logarithms of base-e are called *natural logarithms,* commonly written as "ln x." This section focuses on common logarithms.

To use common logarithms, a table of logarithms (or calculator!) is needed. However, only common logarithms of numbers between 1 and 10 need to be tabulated. Each 1:10 interval is called a *decade*. The part of a logarithm following the decimal point is called the *mantissa.* The whole number part is the *exponent.* Logarithms of numbers less than 1 or greater than 10 are obtained by expressing the number in scientific notation, looking up the significant part of the number b in the table to get the mantissa, and adding the exponent n.

$$\log b \times 10^n = \log b + n \quad \text{Eq. 17}$$

Table 1 Common Logarithms

$N = 10^n$	$N = 10^{-n}$	$n = (-)\log_{10} N$	$N = 10^n$	$N = 10^{-n}$	$n = (-)\log_{10} N$
1.0	(1.0)	0.0000	2.5119	(0.3981)	0.4000
1.1	(0.9091)	0.0414	2.75	(0.3636)	0.4393
1.1111	(0.9000)	0.0458	3.0	(0.3333)	0.4771
1.2	(0.8333)	0.0792	3.1623	(0.3162)	0.5000
1.25	(0.8000)	0.0969	3.3333	(0.3000)	0.5229
1.2589	(0.7943)	0.1000	3.5	(0.2857)	0.5441
1.3	(0.7692)	0.1139	3.9811	(0.2512)	0.6000
1.33 = 4/3	(0.7500)	0.1249	4.0	(0.2500)	0.6020
1.4	(0.7143)	0.1461	4.5	(0.2222)	0.6532
1.4286	(0.7000)	0.1549	5.0	(0.2000)	0.6990
1.5 = 3/2	(0.6667)	0.1761	5.0119	(0.1995)	0.7000
1.5849	(0.6310)	0.2000	5.5	(0.1818)	0.7404
1.6	(0.6250)	0.2041	6.0	(0.1667)	0.7782
1.67 = 5/3	(0.6000)	0.2218	6.3096	(0.1585)	0.8000
1.7	(0.5882)	0.2304	6.5	(0.1538)	0.8129
1.75 = 7/4	(0.5714)	0.2430	7.0	(0.1429)	0.8451
1.8	(0.3652)	0.2553	7.5	(0.1333)	0.8751
1.9	(0.5263)	0.2788	7.9433	(0.1259)	0.9000
1.9953	(0.5012)	0.3000	8.0	(0.1250)	0.9030
2.0	(0.5000)	0.3010	8.5	(0.1176)	0.9294
2.25 = 9/4	(0.4444)	0.3522	9.0	(0.1111)	0.9542
2.5	(0.4000)	0.3979	10.0	(0.1000)	1.0000

Table 1 warrants some explanation. The second column, which is the reciprocal of the number in the first column, is added for convenience. The third column gives the mantissa, the logarithm of the number in the first column. If the logarithm is taken to be a negative value, then it is the mantissa of the number in the second column.

Figure 1 is a graph of $\log x$ and $1/x$. The logarithm of a number less than 1 is negative, as shown by the graph.

To convert a logarithm to a decibel value, multiply by 10. To convert a decibel value to a logarithm, divide by 10. See Eq. 4.

If a level number is greater than 10 or less than 1, express the number in scientific notation, $b \times 10^n$ where b is a number between 1 and 10. Enter b in column 1 and read the mantissa from column 3. Add the exponent n to get the logarithm. See Eq. 17. Convert to decibels by multiplying the logarithm by 10.

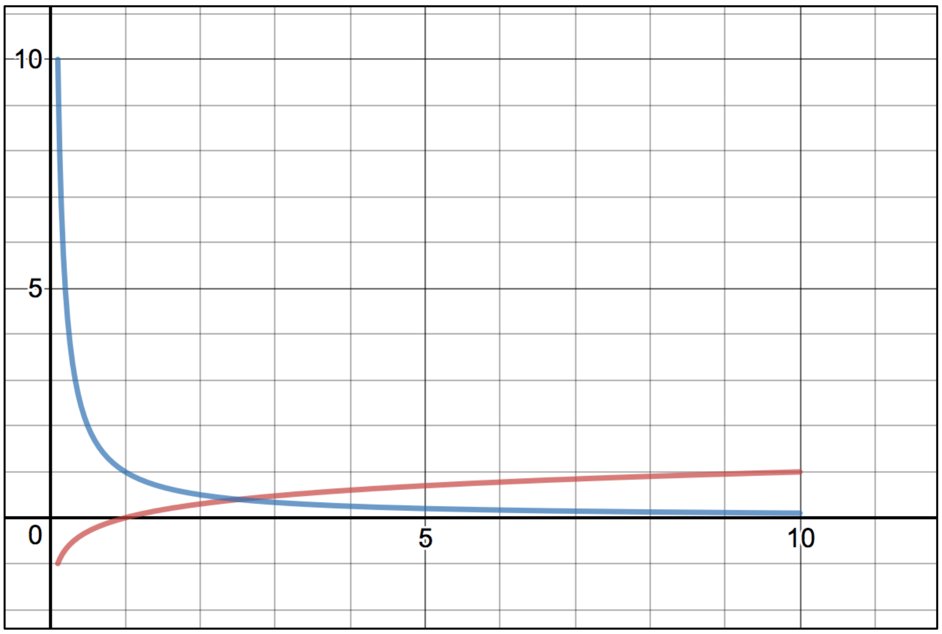

Figure 1 log x (red) and 1/x (blue)

Example 1

What is the loss factor L that corresponds to 1 dB? Enter column 3 with 1/10 = 0.1000 and read the loss factor as 1.2589 from column 1.
The gain is 1.2589 – 1 = 0.2589, or 25.89%. See Eq. 3.

Example 2

What is the loss factor L that corresponds to −1 dB? Infer the minus sign and enter column 3 with 1/10 = 0.1000. Read the loss factor as 0.7943 from column 2.
The loss is 1 – 0.7943 = 0.2057, or 20.57%.

Example 3

What decibel value corresponds of a factor of 2 gain or loss? Enter column 1 with 2.0 and read the logarithm from column 3 as 0.3010. Multiply by 10 to get ±3.010 dB.
+ is a 2× gain, ; – is a ½ loss. Alternatively, express ½ as 5.0×10^{-1}. Enter column 1 with 5.0 and read the mantissa from column 3 as 0.6990. Add the exponent, –1, to get the logarithm, 0.6990–1 = –0.3010. Multiply by 10 to get –3.010 dB.